Comparison of von Zeipel and Modified Hansen Methods of Satellite Theories

~~COMPARISON OF SATELLITE THEORIES~~

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## Abstract

The solutions to the problem of the near earth satellite without drag obtained by applying the von Zeipel method and the modified Hansen method are compared. Differences in the arbitrary constants are tabulated. Transformations are also given relating the time element of the two theories.

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~~COMPARISON OF SATELLITE THEORIES~~

Widely different theories are often used in the computation of orbits of artificial satellites. It is of interest to examine the results of different theories when they are applied to the basic problem of the near earth satellite without drag. Of special importance are the major theories of celestial mechanics introduced by Brouwer (1959) and by Musen (1959), (1961) in solving this problem.

Brouwer (1959) applied the method of von Zeipel to the near earth satellite problem and obtained analytic representations for the osculating Delaunay and Keplerian elements. The results are given by Brouwer to order $J_{20}$ in the elements and $J_{20}^2$ in the mean motions, where $J_{20}$ is the coefficient of the second zonal harmonic of the earth's potential. Musen (1959), (1961) on the other hand, first modified Hansen's method, then by applying it to the same problem of the near earth satellite without drag, showed how to obtain the position of the satellite in a semi-analytic manner to any prescribed order of $J_{20}$. The solution of the satellite problem in terms of orbital true longitude by Musen (1961) is considered below.

The results obtained by Brouwer are given in a form convenient for comparison with the results of many authors. Indeed Kozai (1959), Garfinkel (1959) and others have been able to readily compare their solutions of the satellite problem with Brouwer's solution. However, since Musen's formulations of the problem are intended to provide numerical results of high precision for the position of a satellite, 2 explicit analytic formulations of the perturbations of the elements do not appear in his articles. Consequently, formulas are given below for elements derived from the modified Hansen theory in terms of orbital true longitude so that the results of both authors can be compared.

As one would expect, the differences of the two theories are exhibited in the respective choices of the arbitrary constants and in the arguments of the trigonometric terms. The constants of both theories are discussed and presented in tabular form. The transformations of the variables of the angular arguments are presented. Therefore, when the solutions to the satellite problem are carried out to the same order in $J_{20}$ by the methods of Brouwer and Musen, full correspondence can be obtained by taking into account the differences in the constants and the angular variables.

## The Osculating Elements

The definitions of the osculating elements appearing in Brouwer's article may be found in any text on celestial mechanics, for example, Brouwer and Clemence (1961). It is a relatively simple matter to find expressions for the osculating elements of the modified Hansen theory when expressed in terms of orbital true longitude. These formulas differ from the corresponding formulas of the modified Hansen theory in terms of eccentric anomaly given by Bailie and Bryant (1960) since the W functions differ slightly. We now review briefly some of the concepts and definitions of the modified Hansen theory expressed in terms of orbital true longitude to indicate how representations of osculating elements are derived.

Definitions From the Modified Hansen Theory

When the differential equations given in Musen's article are solved, expressions for the components of the $\overline{W}$ function $\Xi$, $\Upsilon$, and $\Psi$, the $\lambda$ parameters and the perturbation of the pseudo-time $n_0\,\delta z$ result.

The functions $\Xi$, $\Upsilon$, and $\Psi$ are expressed in terms of orbital true longitude and are related to osculating elements by the formulas;

$$\Xi = 1 - \frac{h_0}{h} + 2\frac{h}{h_0}$$

$$\Upsilon = 2\frac{h}{h_0}\, e\cos\phi - \left(1 + \frac{h_0}{h}\right) e_0 \qquad (1)$$

$$\Psi = 2\frac{h}{h_0}\, e\sin\phi$$

Here $-\phi$ is the deviation of the osculating true anomaly from the true anomaly of the auxiliary ellipse, e the osculating eccentricity, and h is proportional to the reciprocal of the Delaunay variable G, that is

$$G = \frac{\mu}{h} \qquad (2)$$

The quantities $h_0$ and $e_0$ are elements of Hansen's auxilary ellipse and are constants.

The $\lambda$ parameters are defined by the formulas;

$$\lambda_1 = \sin\frac{i}{2}\cos N \qquad \lambda_3 = \cos\frac{i}{2}\sin K$$

$$\lambda_2 = \sin\frac{i}{2}\sin N \qquad \lambda_4 = \cos\frac{i}{2}\cos K \tag{3}$$

Here i is the osculating angle of inclination of the orbit plane and corresponds to I in Brouwer's development. The quantities K and N are Fourier series of the order of the perturbations and do not contain secular terms.

The angular variables are given by the formulas;

$$f = cv - \pi_0 - \phi$$

$$\omega = (g-c)v + \left(\pi_0 - \theta_0\right) + \phi + K + N \tag{4}$$

$$\theta = (1-h')v + \theta_0 + K - N$$

The quantities $f$, $\omega$, and $\theta$ are the osculating true anomaly, argument of perigee and longitude of the node. The quantities $g$, $c$, and $h'$ in the right hand side of equations (4) are proportional to the mean motions of the argument of latitude, mean anomaly, and the longitude of the ascending node respectively. The quantities $\pi_0$, and $\theta_0$ are prescribed constants.

The time element of the auxiliary ellipse is denoted by the symbol z and often called the pseudo-time. When orbital true longitude is the 5 argument, the mean anomaly of the auxiliary ellipse is $c(n_0)_H z$. The symbol $n_0$ appears with different meanings in the articles of Brouwer and Musen. Therefore the symbol $(n_0)_H$ is adopted here instead of the $n_0$ appearing in Musen's article. The quantity $\delta z$ is the deviation of the pseudo-time from the unperturbed satellite time.

## Osculating Elements for the Modified Hansen Theory

By inverting equations (1) and (3) it is readily found that

$$G = \frac{\mu}{h} = \frac{\mu}{h_0}\left(1 - \frac{\Xi}{3} + \frac{2}{27}\Xi^2 - \frac{2}{243}\Xi^3 + \cdots\right)$$

$$e = e_0 + \frac{1}{2}\left(\Upsilon - e_0\Xi + \frac{2}{9}e_0\Xi^2 + \frac{3}{16}\frac{\Upsilon^2}{e_0} - \Xi\Upsilon + \frac{\Psi^2}{4e_0} + \cdots\right) \tag{5}$$

$$\cos I = \frac{H}{G} = \cos I_0\left(1 + \frac{\Xi}{3} + \frac{\Xi^2}{27} - \frac{4}{81}\Xi^3 + \cdots\right)$$

Similarly the quantities associated with the angular variables are found to be

$$\phi = \frac{\Psi}{2e_0} + \frac{\Psi}{2e_0}\left[\frac{\Upsilon}{e_0} - \frac{5\Xi}{3}\right] + \cdots$$

$$K + N = \frac{\lambda_3}{\cos\frac{i_0}{2}} + \frac{\lambda_2}{\sin\frac{i_0}{2}} - \frac{\Xi}{12}\cos i_0\left[\frac{\lambda_3}{\cos^2\frac{i_0}{2}} + \frac{\lambda_2}{\sin^2\frac{i_0}{2}}\right] + \cdots \tag{6}$$

$$K - N = \frac{\lambda_3}{\cos\frac{i_0}{2}} - \frac{\lambda_2}{\sin\frac{i_0}{2}} - \frac{\Xi}{12}\cos i_0\left[\frac{\lambda_3}{\cos^2\frac{i_0}{2}} - \frac{\lambda_2}{\sin^2\frac{i_0}{2}}\right] + \cdots$$

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## Comparison of Results to the First Order in $J_{20}$

By solving the equations given in Musen's article, first order analytic solutions for the quantities $\Xi$, $\Upsilon$, $\Psi$, and the $\lambda$ parameters were obtained by Bailie and Fisher (1962). When the analytic expressions for $\Xi$, $\Upsilon$, and $\Psi$ are substituted into equations (5) immediate agreement is obtained with the periodic part of the elements G, e, and I obtained in Brouwer's solution. Similarly, agreement for the periodic part of the expressions for the angular variables $\omega$ and $\theta$ given by equation (4) with the variables g and h can be readily obtained, when the analytic results of Bailie and Fisher are introduced.

It has now been indicated that the periodic part of the solution of the elements of the satellite problem by Brouwer and Musen agree to the first order in $J_{20}$. Although differences in the arbitrary constants and arguments of the trigonometric terms do exist, they do not appear in the first order solutions for the trigonometric parts of the elements since they have $J_{20}$ as a multiplier. These differences are exhibited in the terms of the second order and are discussed below.

## The Arbitrary Constants of the Theories

Differences of order $J_{20}$ appear in the arbitrary constants of the solutions of the satellite problem by Brouwer and by Musen. In a Hansen type theory constants denoted by the symbols $c_0$ and $c_1$ in Musen's article are added to the $\overline{W}$ function and consequently to $\Xi$ and $\Upsilon$. These constants thus occur in the solution for those elements derived from $\Xi$ and $\Upsilon$. In the solution by Brouwer constants appear which represent mean values. In order to compare the two theories the constants to the first order in $J_{20}$ appearing in both theories are listed in Table I.

### Table I

### Constants Appearing in the Satellite Theories (order $J_{20}$)

| Quantity | Brouwer's Solution | Musen's Solution |
|---|---|---|
| $G = \frac{\mu}{h}$ | $G''$ | $\frac{\mu}{h_0}\left(1 - \frac{c_0}{3}\right)$ |
| $e$ | $e'' + \frac{\mu^2 J_{20}(1 - 3\cos I'')}{8e'' G''^4}\left(5 - 3\eta''^2 - 2\eta''^3\right)$ | $e_0 + \frac{c_1 - e_0 c_0}{2}$ |
| $\cos i = \frac{H}{G}$ | $\cos I''$ | $\cos i_0\left(1 + \frac{c_0}{3}\right)$ |
| mean motion of mean anomaly | $\frac{dl''}{dt}$ | $c(n_0)_n$ |

The constants appearing in Table I are defined as follows

$$\eta'' = \sqrt{1 - e''^2}$$

$$\frac{dl''}{dt} = n_0 - \frac{3}{4}\frac{n_0 \mu^2 J_{20}}{L' G''^3}\left(1 - 3\cos^2 I''\right)$$

$$c = 1 + \frac{3}{4} J_{20} \frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right) \tag{7}$$

$$c_0 = \frac{3}{4} J_{20} \frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left(4 - 2\sqrt{1 - e_0^2}\right)$$

$$c_1 = \frac{1}{4}\frac{J_{20}}{e_0}\frac{h_0^4}{\mu^2}\left(1 - 3\cos^2 i_0\right)\left[4\left(1 - \sqrt{1 - e_0^2}\right) + 3e_0^2 - 2e_0^2\sqrt{1 - e_0^2}\right].$$

These values are taken from the article of Brouwer and from the article of Bailie and Fisher.

The relations between the mean motions of the argument of perigee and the longitude of the node in the articles of Brouwer and Musen are given by the formulas

$$\frac{dg''}{dt} = \left(n_0\right)_H (g - c)$$

$$\frac{dh''}{dt} = \left(n_0\right)_H (1 - h') \tag{8}$$

Formulas to order $J_{20}^2$ for these mean motions are given in the article of Brouwer and the article of Bailie and Fisher. At first sight the terms in $J_{20}^2$ seem to disagree. However, by taking the relationships given in Table I into account, full agreement is obtained to order $J_{20}^2$ in the mean motion of the variables as defined in equations (8).

The differences in the constants given in Table I will also be exhibited in the coefficients of trigonometric terms of order $J_{20}^2$ in the elements derived by the methods of Brouwer and of Musen. Additional differences in the coefficients of trigonometric terms of order $J_{20}^2$ appear, due to the differences in the arguments of the trigonometric terms. These are now described.

<u>The Time Elements of the Theories</u>

In the method adopted by Brouwer the true anomalies f and f' appear. Brouwer then shows how to relate these true anomalies to the true time of the satellite. In the method adopted by Musen the true anomaly of the auxiliary ellipse $\bar{f}$ or $\xi$ as it is denoted in the article by Bailie and Fisher appears. Musen shows how to relate $\bar{f}$ and the true time. The true anomalies of the two theories differ by trigonometric terms of the order of $J_{20}$. The relation between these true anomalies is now discussed.

By Taylor's theorem for a function F of $\bar{f}$ we have to the first order in $J_{20}$

$$F(\bar{f}) = F(f) + \frac{\partial F}{\partial \bar{f}}\left(\phi + \frac{\partial f}{\partial e}\,\delta e\right) \tag{8}$$

where $f_{osc} = \bar{f} - \phi$ and $f_{osc}$ is the value of the osculating true anomaly. The quantity f is a function of the osculating mean anomaly $l$, by the equation

$$\frac{df}{dl} = \left(1 - e_0^2\right)^{-3/2} \left(1 + e_0 \cos f\right)$$

also

$$\frac{\partial f}{\partial e} = \left(1 - e_0^2\right)^{-1} \left(2 + e_0 \cos f\right) \sin f$$

The quantity $\delta e$ is the deviation of the osculating eccentricity from the eccentricity of the auxiliary ellipse.

Let us consider the quantity $\delta u = u - \bar{u}$ appearing in Musen's article where $u = 1/r$ and r is the radius vector of the satellite. In equation (8) let us put Fzu, then if one substitutes for $\phi$ and $\delta e$ of equations (8) their values in terms of $\Xi$, T, and $\Psi$ given in equations (4) and recalls that

$$\bar{W} = \bar{\Xi} + \bar{T} \cos \xi + \bar{\Psi} \sin \xi \; ,$$

it is readily found that

$$\delta u = \frac{1}{2} \frac{h_0^2}{\mu} \bar{W} + \frac{\Xi}{6} \bar{u} \; ,$$

to the first order in $J_{20}$, in agreement with the results given in the modified Hansen theory. Thus we have another way of illustrating that $\delta u$ is simply a formula for transforming u as a function of f to u as a function of $\bar{f}$.

Equation (8) does not express $\overline{f}$ as a function of the true time of the satellite. Such a transformation is usually accomplished with the aid of the perturbation of the pseudo-time $\delta z$ .

The perturbation of the pseudo-time is expressed as an infinite series and may converge slowly, particularly for large eccentricities. An alternate method of expressing $\overline{f}$ in terms of true time can readily be found by extending equation (8) thus,

$$F(\overline{f}) = F(f') + \frac{\partial F}{\partial \overline{f}}\left(\phi + \frac{\partial f}{\partial e}\,\delta e + \Delta M\right),$$

to the first order in $J_{20}$. Here $f'$ is the mean true anomaly in the sense given in Brouwer's article and may be evaluated by Kepler's equation for a given value of time.

The perturbation $\Delta M$ is the deviation of the mean anomaly from its mean value. It may be found from the variation equation in terms of orbital true longitude by the methods adopted in the article of Bailie and Fisher.

In particular, if $F(\overline{f}) = \sin\overline{f}$, we have

$$\sin\overline{f} = \sin f' + \cos\overline{f}\left(\phi + \frac{\partial f}{\partial e}\,\Delta e + \Delta M\right).$$

The multiplier of $\cos\overline{f}$ is of order $J_{20}$, so that when $f'$ is given $\overline{f}$ may be found by successive approximations.

## Summary and Conclusions

The solutions to the problem of the near earth satellite without drag given by Brouwer and by Musen agree when carried out to the same order in $J_{20}$. Due allowance must be made for the differences in the constants and in the ways of expressing the time element.

The differences of the arbitrary constants are tabulated to the first order in $J_{20}$. Transformations are given relating the true anomaly of the auxiliary ellipse to the true time of the satellite.

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